Dependence of tropical eigenspaces

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DEPENDENCE OF TROPICAL EIGENSPACES

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Abstract. We study the pathology that causes tropical eigenspaces of distinct supertropical eigenvalues of a non-singular matrix A, to be dependent. We show that in lower dimensions the eigenvectors of distinct eigenvalues are independent, as desired. The index set that differentiates between subsequent essential monomials of the characteristic polynomial, yields an eigenvalue λ, and corresponds to the columns of $adj(A+\lambda I)$ from which the eigenvectors are taken. We ascertain the cause for failure in higher dimensions, and prove that independence of the eigenvectors is recovered in case the "difference criterion" holds, defined in terms of disjoint differences between index sets of subsequent coefficients. We conclude by considering the eigenvectors of the matrix $A^{\nabla} := \frac{1}{det(A)}adj(A)$ and the connection of the independence question to generalized eigenvectors.

1. Introduction

Although supertropical matrix algebra as developed in [20, 21] follows the general lines of classical linear algebra (i.e., a Cayley-Hamilton Theorem, correspondence between the roots of the characteristic polynomial and eigenvalues, Kramer's rule, etc.), one encounters the anomaly in [21, Remark 5.3 and Theorem 5.6] of a matrix whose supertropical eigenvalues are distinct but whose corresponding supertropical eigenspaces are dependent. In this paper we examine how this happens, and give a criterion for the supertropical eigenspaces to be dependent, which we call the **difference criterion**, cf. Definition 3.1 and Theorem 3.4. A pathological example (3.3) is studied in depth to show why the difference criterion is critical. We resolve the difficulty in general in Theorem 3.11 by passing to powers of A and considering generalized supertropical eigenspaces.

1.1. The tropical algebra and related structures.

1.1. The tropical algebra and related structures. We start by discussing briefly the max-plus algebra, its refinements, and their relevance to applications.

The max-plus algebra was inspired by the function log, as the base of the logarithm approaches 0. In the literature, this structure is usually studied via valuations (see [16] and [17]) over the field $K = \mathbb{C}\{\{t\}\}$ of Puiseux series with powers in $\mathbb{Q}$, to the ordered group $(\mathbb{Q}, +, \geq)$. This valuation gives the lowest power of the series (indeed $v(ab) = v(a) + v(b)$ and $v(a + b) \geq \min(v(a), v(b)))$. Then, we look at the dual structure obtained by defining $\text{trop}(a) = -\text{val}(a)$ and denoted as the tropicalization of $a \in K$. By setting $\text{trop}(a+b)$ to be $\max\{\text{trop}(a), \text{trop}(b)\}$, it is obvious that the tropical structure deals with the uncertainty of equality in the valuation, in the form of $\text{trop}(a + a) = \text{trop}(a)$ (also equals to $\text{trop}(-a)$).

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1.2. **The max-plus algebra.**

The **tropical max-plus semifield** is an ordered group $\mathcal{T}$ (usually the additive group of real numbers $\mathbb{R}$ or the set of rational numbers $\mathbb{Q}$), together with a formal element $-\infty$ adjoined. The ordered group $\mathcal{T}$ is made into a semiring equipped with the operations

$$a \oplus b = \max\{a, b\} \quad \text{and} \quad a \odot b = a + b,$$

denoted here as $a + b$ and ab respectively (see [1], [14] and [15]). The unit element $1_{\mathcal{T}}$ is really the element $0 \in \mathbb{Q}$, and $-\infty$ serves as the sero element.

This arithmetic enables one to simplify non-linear questions by answering them in a linear setting (see [13]), which can be applied to discrete mathematics (see [4]), optimization (see [10]) and algebraic geometry (see [14]).

In [12] Gaubert and Sharify introduce a general scaling technique, based on tropical algebra, which applies in particular to the companion form, determining the eigenvalues of a matrix polynomial. Akian, Gaubert and Guterman show in [3] that several decision problems originating from max-plus or tropical convexity are equivalent to zero-sum two player game problems.

[25] is a collection of papers put together by Litvinov and Sergeev. Here, the structure is introduced as a result of the Maslov dequantization applied to traditional mathematics over fields, built on the foundations of idempotent analysis, tropical algebra, and tropical geometry. Applications of idempotent mathematics were introduced by Litvinov and Maslov in [24].

On the pure mathematical side, contributions are made in [25] on idempotent analysis, tropical algebras, tropical linear algebra and tropical convex geometry. Elaborate geometric background with applications to problems in classical (real and complex) geometry can be found in [26]. Here Mikhalkin viewed the tropical structure as a branch of geometry manipulating with certain piecewise-linear objects that take over the role of classical algebraic varieties and describes hypersurfaces, varieties, morphisms and moduli spaces in this setting.

Extensive mathematical applications have been made in combinatorics. In this max-plus language, we may use notions of linear-algebra to interpret combinatorial problems. In [23] Jonczy presents some problems described by the Path algebra and solved by means of min and max operations. The main combinatorial surveys are [7], [8] of Butkovic and [9] of Butkovic and Murfitt, which focus on presenting a number of links between basic max-algebraic problems on the one hand and combinatorial problems on the other hand. This indicates that the max-algebra may be regarded as a linear-algebraic encoding of a class of combinatorial problems.

1.3. **Supertropical algebra.** We pass to a cover of the max-plus semifield, called the **supertropical semiring**, equipped with the ghost ideal $\mathcal{G} := \mathcal{T}^{\nu}$, as established and studied by Izhakian and Rowen in [18] and [19].

We denote as $R = \mathcal{T} \cup \mathcal{G} \cup \{-\infty\}$ the "standard" supertropical semiring, which contains the so-called tangible elements of the structure and where we have a projection $R \to \mathcal{G}$ given by $a \mapsto a^{\nu}$ for $a \in \mathcal{T}$. $\{a^{\nu} \in \mathcal{G}, \forall a \in \mathcal{T}\}$ are the ghost elements of the structure, as defined in [19]. So $\mathcal{G}$ inherits the order of $\mathcal{T}$. We write 0_R for $-\infty$, to stress its role as the zero element.

This enables us to distinguish between a maximal element a that is attained only once in a sum, i.e., $a \in \mathcal{T}$ which is invertible, and a maximum that is being attained at least twice, i.e., $a+a = a^\nu \in \mathcal{G}$, which is not invertible. We do not distinguish between $a+a$ and $a+a+a$ in this structure. Note that ν projects the standard supertropical semiring onto $\mathcal{G}$, which can be identified with the usual tropical structure.

In this new supertropical sense, we use the following order relation to describe two elements that are equal up to a ghost supplement:

Definition 1.1. Let a, b be any two elements in R. We say that a **ghost surpasses** b, denoted $a \models_{gs} b$, if $a = b + ghost$. That is, $a = b$ or $a \in \mathcal{G}$ with $a^\nu \geq b^\nu$.

We say a is ν-equivalent to b, denoted by $a \cong_\nu b$, if $a^\nu = b^\nu$. That is, in the tropical structure, ν-equivalence projects to equality.

We adjust these notations for matrices (and in particular for vectors) entry-wise, and for polynomials coefficient-wise.

Important properties of $\models_{gs}$:

(1) $\models_{gs}$ is a partial order relation (see [21, Lemma 1.5]).
(2) If $a \models_{gs} b$ then $ac \models_{gs} bc$.
(3) If $a \models_{gs} b$ and $c \models_{gs} d$ then $a + c \models_{gs} b + d$ and $ac \models_{gs} bd$.

Considering this relation, we regain basic algebraic properties that were not accessible in the usual tropical setting, such as multiplicativity of the tropical determinant, the near multiplicativity of the tropical adjoint, the role of roots in the factorization of polynomials, the role of the determinant in matrix singularity, a matrix that acts like an inverse, common behavior of similar matrices, classical properties of $\operatorname{adj}(A)$, and the use of elementary matrices. Tropical eigenspaces and their dependences are of considerable interest, as one can see in [2], [5], [7], [18], [21] and [29].

Many of these properties will be formulated in the Preliminaries section. We would also like to attain a supertropical analog to eigenspace decomposition, but we encounter the example of [21]. Our objective in this paper is to understand how such an example arises, and how it can be circumvented, either by introduce the **difference criterion** of Definition 3.1 or by passing to generalized eigenspaces in § 3.3.3.

2. Preliminaries

In this section, we introduce the notations, definitions and fundamental results to be used throughout this paper. We present tropical polynomials, together with well-known and recent results. Then we introduce definitions and relevant properties of matrices and vectors in the tropical structure. We introduce extended definitions within the supertropical framework and demonstrate their use in the tropical framework.

2.1. **Tropical Polynomials.**

Notation 2.1.

Throughout, for an element $a \in R$, we consider a^ν to be the element $b \in \mathcal{G}$ s.t. $a \cong_\nu b$, and $\hat{a}$ to be the element $b \in \mathcal{T}$ s.t. $a \cong_\nu b$. (We define $0_R^\nu = \widehat{0_R} = 0_R$.)
We adjust these notations for matrices (and in particular for vectors) entry-wise, and for polynomials coefficient-wise.

Definition 2.2. Let $k \in \mathbb{N}$. Defining $b = a^k$ to be the tropical product of a by itself k times (i.e., $a^{\odot k} = a \odot \cdots \odot a = a + \cdots + a = ka$), we may consider that a is a k-**root** of b, denoted as $a = \sqrt[k]{b}$. This operation is well-defined on $\mathcal{T}$.

Clearly, any tropical polynomial takes the value of the dominant monomial along the $\mathcal{T}$-axis. That having been said, it is possible that some monomials in the polynomial would not dominate for any $x \in \mathcal{T}$.

Definition 2.3. Let $f(x) = \sum_{i=0}^{n} \alpha_i x^{n-i} \in R[x]$ be a tropical polynomial. We call monomials in $f(x)$ that dominate for some $x \in R$ **essential**, and monomials in $f(x)$ that do not dominate for any $x \in R$ **inessential**. We write $f^{es}(x) = \sum_{k \in I} \alpha_k x^{n-k} \in R[x]$, where $\alpha_k x^{n-k}$ is an essential monomial $\forall k \in I$, called **the essential polynomial** of f.

In the classical sense, a root of a tropical polynomial can only be 0_R, which occurs if and only if the polynomial has constant term 0_R. We would like the roots to indicate the factorization of the polynomial, which leads to the following tropical definition of a root.

Definition 2.4. We define an element $r \in R$ to be a **root** of a tropical polynomial $f(x)$ if $f(r) \models_{gs} 0_R$.

We refer to roots of a polynomial being obtained as a simultaneous value of two leading tangible monomials as **corner roots**, and to roots that are being obtained from one leading ghost monomial as **non-corner roots**. We factor polynomials viewing them as functions. Then, for every corner root r of f, we may write f as $(x+r)^k g(x)$ for some $g(x) \in R[x]$ and $k \in \mathbb{N}$, where k is the difference between the exponents of the tangible essential monomials attaining r.

2.2. Matrices.

As defined over a ring, for matrices $A = (a_{i,j}) \in M_{n\times m}(R),\ B = (b_{i,j}) \in M_{s\times t}(R)$

$$\begin{cases} A + B = (c_{i,j}) :\ c_{i,j} = a_{i,j} + b_{i,j} & ,\ \text{defined iff } n = s, m = t\ , \\ \\ AB = (d_{i,j})\ :\ d_{i,j} = \sum_{k=1}^{n} a_{i,k} b_{k,j} & ,\ \text{defined iff }\ m = s\ . \end{cases}$$

Definition 2.5. Let $\pi \in S_n$ and $A = (a_{i,j}) \in M_n(R)$. The **permutation** π **of A** is the word

$$a_{1,\pi(1)} a_{2,\pi(2)} \cdots a_{n,\pi(n)}.$$

The word $a_{1,1}a_{2,2}\cdots a_{n,n}$ is denoted as the **identity or Id-permutation**, corresponding to the diagonal of A. We write a permutation of A as a product of disjoint cycles $C_1, \dots, C_t$, where $\{C_i\}$ corresponds to the disjoint cycles composing π.

We define the tropical **trace** and **determinant** of A to be

$$tr(A) = \sum_{k=1}^{n} a_{k,k} \quad \text{and} \quad \det(A) = \sum_{\sigma \in S_n} a_{1,\sigma(1)} \cdots a_{n,\sigma(n)},$$

respectively.

In the special case where $A \in M_n(R)$, we refer to any entry attaining the trace as **a dominant diagonal entry**. We call $a_{1,\sigma(1)} \cdots a_{n,\sigma(n)}$ the **weight** contributed by σ to the determinant, and any permutation whose weight has the same ν-value as the determinant is **a dominant permutation of A**.

If there is a single dominant permutation, its weight equals the determinant.

Unlike over a field, the tropical concepts of singularity, invertability and factorizability do not coincide. We would like the determinant to indicate the singularity of a matrix. Hence, we define a matrix $A \in \mathrm{M}_n(R)$ to be **tropically singular** if there exist at least two different dominant permutations. Otherwise the matrix is **tropically nonsingular**. Consequently, a matrix $A \in \mathrm{M}_n(R)$ is supertropically singular if $\det(A) \models_{gs} 0_R$ and supertropically nonsingular if $\det(A) \in \mathcal{T}$. A matrix A is **strictly singular** if $\det(A) = 0_R$.

A surprising result in this context is that the product of two nonsingular matrices might be singular, but we do have:

Theorem 2.6. *For $n \times n$ matrices A, B over the supertropical semiring R, we have*

$$\det(AB) \models_{gs} \det(A)\det(B).$$

This theorem has been proved in [20, Theorem 3.5] due to considerations of graph theory, but also in [11, Proposition 2.1.7] by using the transfer principles (see [2, Theorem 3.3 and Theorem 3.4]). These theorems allow one to obtain such results automatically in a wider class of semirings, including the supertropical semiring.

Definition 2.7. Suppose $\mathcal{R}$ is a semiring. An $\mathcal{R}$**-module** V is a semigroup $(V, +, 0_V)$ together with scalar multiplication $\mathcal{R} \times V \to V$ satisfying the following properties for all $r_i \in \mathcal{R}$ and $v, w \in V$:

(1) $r(v + w) = rv + rw$
(2) $(r_1 + r_2)v = r_1v + r_2v$
(3) $(r_1r_2)v = r_1(r_2v)$
(4) $1_{\mathcal{R}}v = v$
(5) $r \cdot 0_V = 0_V$
(6) $0_{\mathcal{R}} \cdot v = 0_V$.

For any semiring R, let R^n be the free module of rank n over R. We define the **standard base** to be $e_1, \dots, e_n$, where

$$e_i = \begin{cases} 1_{\mathcal{T}} = 1_R, & \text{in the } i^{th} \text{ coordinate} \\ 0_{\mathcal{T}} = 0_R, & \text{otherwise} \end{cases}.$$

The tropical **identity matrix** is the $n \times n$ matrix with the standard base for its columns. We denote this matrix as $I_{\mathcal{T}} = I$.

A matrix $A \in \mathrm{M}_n(R)$ is **invertible** if there exists a matrix $B \in \mathrm{M}_n(R)$ such that $AB = BA = I$.

From now on $\mathcal{F} := \mathcal{T} \cup \mathcal{G} \cup \{0_{\mathcal{F}}\}$, where its set $\mathcal{T}$ is presumed to be a group, and $\mathcal{G}$ is its ghost elements. We write $V = \mathcal{F}^n$, with the standard base $\{e_1, \dots, e_n\}$.

Definition 2.8. We define vectors $v_1, \dots, v_k$ in V to be **tropically dependent** if there exist $a_1, \dots, a_k \in \mathcal{T}$ such that $\sum_{i=1}^{k} a_iv_i \models_{gs} \overrightarrow{0_{\mathcal{F}}}$. Otherwise, we define this set of tropical vectors to be **independent**.

We define two types of special matrices:

Definition 2.9. An $n \times n$ matrix $P = (p_{i,j})$ is a **permutation matrix** if there exists $\pi \in S_n$ such that

$$p_{i,j} = \begin{cases} 0_{\mathcal{F}}, & j \neq \pi(i) \\ 1_{\mathcal{F}}, & j = \pi(i) \end{cases}.$$

Since $\forall \pi \in S_n \ \exists! \sigma \in S_n : \ \sigma = \pi^{-1}$ and $1_{\mathcal{F}}$ is invertible, a permutation matrix is always invertible.
An $n \times n$ matrix $D = (d_{i,j})$ is a **diagonal matrix** if

$$\exists \ a_1, \dots, a_n \in \mathcal{F} : \ d_{i,j} = \begin{cases} 0_{\mathcal{F}}, & j \neq i \\ a_i, & j = i \end{cases},$$

which is invertible if and only if $\det(D)$ is invertible (i.e., $a_i \in \mathcal{T}, \ \forall i$).

Remark 2.10. (See [20, Proposition 3.9]) A tropical matrix A is invertible if and only if it is a product of a permutation matrix and an invertible diagonal matrix. These types of products are called **generalized permutation matrices**, that is $(d_{i,j})$ such that

$$\exists \ a_1, \dots, a_n \in \mathcal{T}, \ \pi \in S_n : \ d_{i,j} = \begin{cases} 0_{\mathcal{F}}, & j \neq \pi(i) \\ a_i, & j = \pi(i) \end{cases}.$$

We define three types of tropical elementary matrices, corresponding to the three elementary matrix operations, obtained by applying one such operation to the identity matrix.

A transposition matrix is obtained from the identity matrix by switching two rows (resp. columns). This matrix is invertible: $E_{i,j}^{-1} = E_{i,j}$, and a product of transposition matrices yields a permutation matrix.

An elementary diagonal multiplier is obtained from the identity matrix where one row (resp. column) has been multiplied by an invertible scalar. This matrix is invertible: $E^{-1}_{\alpha \cdot i^{th} row} = E_{\alpha^{-1} \cdot i^{th} row}$, and a product of diagonal multipliers yields an invertible diagonal matrix.

A Gaussian matrix is defined to differ from the identity matrix by having a non-zero entry in a non-diagonal position. We denote as $E_{i^{th} row + \alpha \cdot j^{th} row}$ the elementary Gaussian matrix adding row j, multiplied by α, to row i. By Remark 2.10, this matrix is not invertible.

2.2.1. *The supertropical approach.* Having established that algebraically $\mathcal{G} \cup \{-\infty\}$ and $\models_{gs}$ effectively take the role of singularity and equality over $\mathcal{F}$, we would like to extend additional definitions to the supertropical setting, using ghosts for zero.
A **quasi-zero** matrix $Z_{\mathcal{G}}$ is a matrix equal to $0_{\mathcal{F}}$ on the diagonal, and whose off-diagonal entries are ghost or $0_{\mathcal{F}}$.
A **diagonally dominant** matrix is a non-singular matrix with a dominant permutation along the diagonal.
A **quasi diagonally dominant** matrix $D_{\mathcal{G}}$ is a diagonally dominant matrix A whose off-diagonal entries are ghost or $0_{\mathcal{F}}$.
A **quasi-identity** matrix $I_{\mathcal{G}}$ is a nonsingular, multiplicatively idempotent matrix equal to $I + Z_{\mathcal{G}}$, where $Z_{\mathcal{G}}$ is a quasi-zero matrix.

Thus, every quasi-identity matrix $I_{\mathcal{G}}$ is quasi diagonally dominant. Using the tropical determinant, we attain the tropical analog for the well-known *adjoint*.

Definition 2.11. The r,c-**minor** $A_{r,c}$ of a matrix $A=(a_{i,j})$ is obtained by deleting row r and column c of A. The **adjoint matrix** $\mathrm{adj}(A)$ of A is defined as the matrix $(a'_{i,j})$, where $a'_{i,j}=\det(A_{j,i})$. When $\det(A)$ is invertible, the matrix A^{∇} denotes

$$\frac{1}{\det(A)}\,\mathrm{adj}(A).$$

Notice that $\det(A_{j,i})$ may be obtained as the sum of all permutations in A passing through $a_{j,i}$, but with $a_{j,i}$ deleted:

$$\det(A_{j,i})=\sum_{\substack{\sigma\in S_n:\\ \sigma(j)=i}} a_{1,\sigma(1)}\cdots a_{j-1,\sigma(j-1)}a_{j+1,\sigma(j+1)}\cdots a_{n,\sigma(n)}.$$

When writing each permutation as the product of disjoint cycles, $\det(A_{j,i})$ can be presented as:

$$\det(A_{j,i})=\sum_{\substack{\sigma\in S_n:\\ \sigma(j)=i}}(a_{i,\sigma(i)}\cdots a_{\sigma^{-1}(j),j})C_{\sigma},$$

where $(a_{i,\sigma(i)}\cdots a_{\sigma^{-1}(j),j})$ is the cycle with $a_{j,i}$ deleted, and C_σ is the product of the cycles of σ that do not include i and j.

Definition 2.12. We say that A^{∇} is the **quasi-inverse** of A over $\mathcal{F}$, denoting

$$I_A=AA^{\nabla}\text{ and }I'_A=A^{\nabla}A,$$

where I_A,I'_A are quasi-identities (see [21, Theorem 2.8]).

As a result of these supertropical definitions, we gain a tropical version for two well-known algebraic properties, proved in Proposition 4.8. and Theorem 4.9. of [20].

Proposition 2.13. $\mathrm{adj}(AB)\models_{gs}\mathrm{adj}(B)\,\mathrm{adj}(A)$.

As a result, one concludes from the third property of $\models_{gs}$ (see Definition 1.1) that

$$(AB)^{\nabla}\models_{gs}B^{\nabla}A^{\nabla}.$$

Theorem 2.14.

(i) $\det(A\cdot\mathrm{adj}(A))=\det(A)^n$.

(ii) $\det(\mathrm{adj}(A))=\det(A)^{n-1}$.

Remark 2.15. (see [28, Remark 2.18]) For a definite matrix A we have

$$A^{\nabla}=\frac{1}{\det(A)}\,\mathrm{adj}(A)=\mathrm{adj}(A),$$

which is also definite.

The following lemma has been proved in [28, Lemma 3.2], and states the connection between multiplicity of the determinant and the quasi-inverse matrix:

Lemma 2.16. *Let P be an invertible matrix and A be non-singular.*

(i) $P^\nabla = P^{-1}$.

(ii) $\det(PA) = \det(P)\det(A)$.

(iii) $(PA)^\nabla = A^\nabla P^\nabla$ *(that is, equality holds in Proposition 2.13).*

(iv) *If* $A = P\bar{A}$*, where* $\bar{A}$ *is the definite form of* A *with left normalizer* P*, then* $A^\nabla = \bar{A}^\nabla P^{-1}$ *where* $\bar{A}^\nabla$ *is definite, with right normalizer* P^{-1}*.*

Matrix invariants

Let $A \in \mathrm{M}_n(\mathcal{F})$. We continue the supertropical approach by defining $v \in V$, not all singular, such that $\exists \lambda \in \mathcal{T} \cup \{0_{\mathcal{F}}\}$ where $Av \models_{gs} \lambda v$, to be a **supertropical eigenvector** of A with a **supertropical eigenvalue** λ, having an **eigenmatrix** $A + \lambda I$. The **characteristic polynomial** of A (also called the maxpolynomial, cf.[8]) is defined to be

$$f_A(x) = \det(xI + A).$$

The tangible value of its roots are the eigenvalues of A, as shown in [20, Theorem 7.10]. Following to Definition 2.4, we may have *corner eigenvalues* and *non-corner eigenvalues.*

The coefficient of x^{n-k} in this polynomial is the sum of determinants of all $k \times k$ **principal sub-matrices**, otherwise known as the trace of the k^{th} compound matrix of A. Thus, this coefficient, which we denote as α_k, takes the dominant value among the permutations on all subsets of indices of size k:

$$\alpha_k = \sum_{\substack{I \subseteq [n]: \\ |I| = k}} \sum_{\sigma \in S_k} \prod_{i \in I} a_{i,\sigma(i)}.$$

When $\alpha_k \in \mathcal{T}$, we define the **index set of** α_k, denoted by Ind_k, a set $I \subseteq [n]$ on which the dominant permutation defining α_k is obtained.

Let $f_A(x) = \sum_{i=0}^{n} \alpha_i x^{n-i}$ be the characteristic polynomial of A, with the essential polynomial

$$f_A^{es}(x) = \sum_k \alpha_{i_k} x^{n-i_k}.$$

Let λ be the corner eigenvalue obtained between the essential monomial $\alpha_{i_{k-1}} x^{n-(i_{k-1})}$ and the subsequent essential monomial $\alpha_{i_k} x^{n-i_k}$. We denote $I_\lambda = \mathrm{Ind}_{i_k} \setminus \mathrm{Ind}_{i_{k-1}}$.

Theorem 2.17. *(The eigenvectors algorithm, see* [21, Remark 5.3 and Theorem 5.6]*.) Let* $t \in I_\lambda$*. The tangible value of the* t^{th}*-column of* $\mathrm{adj}(\lambda I + A)$ *is a tropical eigenvector of* A *with respect to the eigenvalue* λ*.*

This algorithm will be demonstrated in §3.2.

The Supertropical Cayley-Hamilton Theorem has been proved in [20, Theorem 5.2], and is as follows:

Theorem 2.18. *Any matrix* A *satisfies its tangible characteristic polynomial* f_A*, in the sense that* $f_A(A)$ *is ghost.*

One can find a combinatorial proof in [30] and a proof using the transfer principle in [2].

In analogy to the classical theory, we have

Proposition 2.19. *([20, Proposition 7.7]) The roots of the polynomial $f_A(x)$ are precisely the supertropical eigenvalues of A.*

Remark 2.20. Recall that a supertropical polynomial is r-**primary** if it has the unique supertropical root r. It is well-known that any tropical r-primary polynomial has the form $(x+r)^m$ for some $m \in \mathbb{N}$, and any tropical essential polynomial f_A can be factored as a function to a product of primary polynomials, and thus of the form $\prod_i g_i$ where $g_i = (x+r_i)^{m_i}$. The supertropical version of this is given in [19, Theorem 8.25 and Theorem 8.35].

Another classical property attained in this extended structure is:

Proposition 2.21. *If $\lambda \in \mathcal{T} \cup \{0_{\mathcal{F}}\}$ is a supertropical eigenvalue of a matrix $A \in \mathrm{M}_n(\mathcal{F})$ with eigenvector v, then λ^i is a supertropical eigenvalue of A^i, for every $i \in \mathbb{N}$, with respect to the same eigenvector.*

Theorem 2.22. *Let A be a non-singular matrix.*

(1) ([27, Theorem 3.6]) For any $m \in \mathbb{N}$ we have

$$f_{A^m}(x^m) \models_{gs} (f_A(x))^m,$$

implying that the m^{th}-root of every corner eigenvalue of A^m is a corner eigenvalue of A.

(2) ([6, Theorem 5.1]) For A^{∇}, the quasi-inverse of A, we have

$$det(A) f_{A^{\nabla}}(x) \models_{gs} x^n f_A(x^{-1}),$$

implying that the inverse of every corner eigenvalue of A^{∇} is a corner eigenvalue of A.

3. Dependence of eigenvectors

A well-known decomposition of F^n, where F is a field, is the decomposition to eigenspaces of a matrix $A \in \mathrm{M}_n(F)$. In particular, this decomposition is obtained when the eigenvalues are distinct since, in the classical case, eigenspaces of distinct eigenvalues are linearly independent, which compose a basis for F^n. In the tropical case, considering that dependence occurs when a tropical linear combination ghost-surpasses $\overrightarrow{0_{\mathcal{F}}}$, such a property need not necessarily hold.

In the upcoming section we analyze the dependence between eigenvectors, using their definition according to the algorithm described in Theorem 2.17. We present special cases in which this undesired dependence is resolved.

Definition 3.1. The matrix A satisfies the **difference criterion** if the sets I_λ, such that λ is a corner root of f_A, are disjoint.

3.1. Eigenspaces in lower dimensions.

In the following proposition, we verify that independence of eigenvectors, of distinct eigenvalues, holds in dimension 2 and 3.

Proposition 3.2. *Let $A = (a_{i,j})$ be an $n \times n$ nonsingular matrix, where $n \in \{2, 3\}$, with a tangible characteristic polynomial (coefficient-wise) and n distinct eigenvalues. Then the eigenvectors of A are tropically independent.*

Proof.
The 2×2 case:
Let $f_A(x) = x^2 + tr(A)x + \det(A)$ be the characteristic polynomial of A. If A has two distinct eigenvalues, then these must be $\lambda_1 = tr(A)$ and $\lambda_2 = \frac{\det(A)}{tr(A)}$.

We must have $\lambda_1 > \lambda_2$, for otherwise either

$$f_A(\lambda_2) = \frac{\det(A)}{tr(A)} \left(\frac{\det(A)}{tr(A)} + tr(A)^\nu \right) = \left(\frac{\det(A)}{tr(A)} \right)^2 \in \mathcal{T},$$

or $\lambda_1 = \lambda_2$, which means the polynomial has one root with multiplicity 2.
Without loss of generality, we may assume that $tr(A) = a_{1,1}$. According to the algorithm, since $I_{\lambda_1} = \{1\}$, λ_1 has the eigenvector obtained by the tangible value of the *first* column of its eigenmatrix. Since $I_{\lambda_2} = \{2\}$, λ_2 has the eigenvector obtained by the tangible value of the *second* column of its eigenmatrix.
The determinant is either:

$$\det(A) = a_{1,1}a_{2,2}, \text{ where } a_{1,1} > a_{2,2} \text{ and } a_{1,1}a_{2,2} > a_{1,2}a_{2,1},$$

(and then the eigenvalues are $a_{1,1}$ and $a_{2,2}$,) or

$$\det(A) = a_{1,2}a_{2,1}, \text{ where } a_{1,1}a_{2,2} < a_{1,2}a_{2,1},$$

(and then the eigenvalues are $a_{1,1}$ and $\frac{a_{1,2}a_{2,1}}{a_{1,1}}$, satisfying $a_{1,1} > \frac{a_{1,2}a_{2,1}}{a_{1,1}} > a_{2,2}$).
In both cases, the first column of $\operatorname{adj}(A + \lambda_1 I)$ is $(a_{1,1}, a_{2,1})$ and the second column of $\operatorname{adj}(A + \lambda_2 I)$ is $(a_{1,2}, a_{1,1})$, which are tropically independent since $a_{1,1}^2 > a_{1,2}a_{2,1}$.

The 3×3 case:

This case indicates key techniques for understanding and motivating the general proof on matrices satisfying the difference criterion in §3.3.1.
Let

$$f_A(x) = \sum_{i=0}^{3} a_i x^{3-i} = x^3 + tr(A)x^2 + \alpha x + \det(A)$$

be the characteristic polynomial of A. We assign $tr(A)$ to be $a_{1,1}$, i.e.,

$$a_{1,1} > a_{t,t} \ \forall t \neq 1. \tag{3.1}$$

For the determinant we have six permutations of S_3. In order to obtain three distinct eigenvalues, we must have

$$\lambda_1 = tr(A) > \lambda_2 = \frac{\alpha}{tr(A)} > \lambda_3 = \frac{\det(A)}{\alpha}, \tag{3.2}$$

for otherwise $\exists t, s : \ f_A(\lambda_t) \in \mathcal{T}$ or $\lambda_t = \lambda_s$. Thus

$$\lambda_1\lambda_2 = \alpha \ \text{ and } \ \lambda_1\lambda_2\lambda_3 = \det(A). \tag{3.3}$$

As a result, $\mathrm{Ind}_1 \subseteq \mathrm{Ind}_2$; otherwise, $a_{1,1}$ together with α yields a permutation whose weight is dominated by $\det(A)$, and we get $\lambda_1 = a_{1,1} < \frac{\det(A) \cdot a_{1,1}}{\alpha \cdot a_{1,1}} = \lambda_3$, contrary to (3.2).

Therefore,

$$\begin{cases} I_{\lambda_1} = \{1\} \setminus \emptyset = \{1\} \\ I_{\lambda_2} = \{1, j\} \setminus \{1\} = \{j\}, \\ I_{\lambda_3} = \{1, j, k\} \setminus \{1, j\} = \{k\}, \end{cases}$$

where $1, j, k$ are distinct. Without loss of generality, we may take $j = 2$ and $k = 3$, and obtain the eigenmatrices:

$$A + \lambda_1 I = \begin{pmatrix} \lambda_1 & a_{1,2} & a_{1,3} \\ a_{2,1} & \lambda_1 & a_{2,3} \\ a_{3,1} & a_{3,2} & \lambda_1 \end{pmatrix} \text{ (since } a_{1,1} > a_{t,t} \quad \forall t \neq 1\text{)},$$

$$A + \lambda_2 I = \begin{pmatrix} \lambda_1 & a_{1,2} & a_{1,3} \\ a_{2,1} & \lambda_2 & a_{2,3} \\ a_{3,1} & a_{3,2} & \lambda_2 \end{pmatrix} \text{ (since } \frac{\alpha \cdot a_{t,t}}{tr(A) \cdot a_{t,t}} \geq a_{t,t} \quad \forall t \neq 1\text{)}$$

and

$$A + \lambda_3 I = \begin{pmatrix} \lambda_1 & a_{1,2} & a_{1,3} \\ a_{2,1} & \beta & a_{2,3} \\ a_{3,1} & a_{3,2} & \lambda_3 \end{pmatrix} \text{ (since } \frac{\det(A) \cdot a_{3,3}}{\alpha \cdot a_{3,3}} \geq a_{3,3}\text{)},$$

where $\beta = \max\{a_{2,2}, \lambda_3\}$.

Recalling the algorithm in Theorem 2.17, we let W be the matrix with the (tangible value of the) eigenvectors for its columns

$$W = \begin{pmatrix} \lambda_1^2 & a_{1,2}\lambda_2 + a_{1,3}a_{3,2} & a_{1,3}\alpha + a_{1,2}a_{2,3} \\ \\ a_{2,1}\lambda_1 + a_{2,3}a_{3,1} & \lambda_1\lambda_2 & a_{2,3}\lambda_1 + a_{2,1}a_{1,3} \\ \\ a_{3,1}\lambda_1 + a_{3,2}a_{2,1} & a_{3,2}\lambda_1 + a_{3,1}a_{1,2} & \lambda_1\beta + a_{1,2}a_{2,1} \end{pmatrix},$$

Furthermore, the $(3, 3)$ position is $\alpha = \lambda_1\lambda_2$, since $\lambda_2 > \lambda_3$:

$$\begin{cases} \text{If } \alpha = a_{1,2}a_{2,1}, & \text{then both } \lambda_1 a_{2,2} < \alpha \text{ and } \lambda_1\lambda_3 < \alpha, \text{ implying } \lambda_1\beta + a_{1,2}a_{2,1} = \alpha. \\ \text{If } \alpha = a_{1,1}a_{2,2}, & \text{then } \lambda_3 < \frac{a_{2,2}}{a_{1,1}} a_{1,1} = \lambda_2, \text{ so } \beta = a_{2,2} \text{ and } \lambda_1 a_{2,2} + a_{1,2}a_{2,1} = \alpha. \end{cases}$$

Due to relations (3.1), (3.2) and (3.3), $\det(W) = \lambda_1^2(\lambda_1\lambda_2)(\lambda_1\lambda_2)$ is obtained solely from the diagonal and therefore is tangible. (We further elaborate this statement in the generalization proved in Theorem 3.4.)

□

3.2. **The pathology appears.**

We follow Example 3.3, introduced in [21], to show how independence of eigenspaces might fail for dimensions higher then 3, due to the increased variety of indices. While applying the eigenvectors-algorithm, we demonstrate how the classical algorithm for finding the eigenvectors by reduction of the eigenmatrix, still holds in the tropical case, when treating the ghosts as "zero-elements". This illustrative example will provide the motivation for Theorem 3.4, Conjecture 3.5 and Conjecture 3.6, generalizing the connection of the index sets to the dependence of the eigenvectors.

Example 3.3. Let

$$A = \begin{pmatrix} 10 & 10 & 9 & - \\ 9 & 1 & - & - \\ - & - & - & 9 \\ 9 & - & - & - \end{pmatrix}.$$

The characteristic polynomial of A is

$$f_A(x) = x^4 + 10x^3 + 19x^2 + 27x + 28,$$

obtained from the permutations (1), (1 2), (1 3 4), (1 3 4)(2), respectively. Therefore,

$$\begin{cases} I_{\lambda_1} = \{1\} \setminus \emptyset = \{1\}, \\ I_{\lambda_2} = \{1,2\} \setminus \{1\} = \{2\}, \\ I_{\lambda_3} = \{1,3,4\} \setminus \{1,2\} = \{3,4\}, \\ I_{\lambda_4} = \{1,2,3,4\} \setminus \{1,3,4\} = \{2\} \end{cases} \tag{3.4}$$

where $\lambda_1 = 10,\ \lambda_2 = 9,\ \lambda_3 = 8$ and $\lambda_4 = 1$, are the eigenvalues of A. As we saw in §3.1, the overlap of the second and fourth sets cannot occur in lower dimensions.
The eigenmatrices and eigenvectors are as follows:

<u>For λ_1 :</u>

$$A + 10I = \begin{pmatrix} 10^\nu & 10 & 9 & - \\ 9 & 10 & - & - \\ - & - & 10 & 9 \\ 9 & - & - & 10 \end{pmatrix},$$

and the tangible value of the first column of its adjoint is

$$v_1 = (30, 29, 28, 29) = 28\,(2, 1, 0, 1)\,.$$

This can also be obtained when multiplying the eigenmatrix by

$$E^2_{4^{th}\ row + 1\cdot 3^{rd}\ row} E_{4^{th}\ row + 1\cdot 2^{nd}\ row} E_{2^{nd}\ row + 1^{st}\ row} E_{1,4}$$

on the left:

$$\begin{pmatrix} 9 & - & - & 10 \\ 9^\nu & 10 & - & 10 \\ - & - & 10 & 9 \\ 10^\nu & 10^\nu & 12^\nu & 11^\nu \end{pmatrix},$$

and solving the tropically linear system

$$\begin{cases} 9x + 10w \in \mathcal{G}, \\ 10y + 10w \in \mathcal{G}, \\ 10z + 9w \in \mathcal{G}, \end{cases}$$

which yields $(11, 10, 9, 10) = 9\,(2, 1, 0, 1)\,,$ a multiple of v_1.

<u>For λ_2 :</u>

$$A + 9I = \begin{pmatrix} 10 & 10 & 9 & - \\ 9 & 9 & - & - \\ - & - & 9 & 9 \\ 9 & - & - & 9 \end{pmatrix},$$

and the tangible value of the second column of its adjoint is

$$v_2 = (28, 28, 28, 28) = 28\,(0, 0, 0, 0)\,.$$

This can also be obtained when multiplying the eigenmatrix by

$$E_{4^{th}\ row + 2\cdot 3^{rd}\ row} E_{4^{th}\ row + 1\cdot 2^{nd}\ row} E_{2^{nd}\ row + 1^{st}\ row} E_{1,4}$$

on the left:

$$\begin{pmatrix} 9 & - & - & 9 \\ 9^{\nu} & 9 & - & 9 \\ - & - & 10 & 9 \\ 10^{\nu} & 10^{\nu} & 9^{\nu} & 9^{\nu} \end{pmatrix},$$

and solving the tropically linear system

$$\begin{cases} 9x + 9w \in \mathcal{G}, \\ 9y + 9w \in \mathcal{G}, \\ 9z + 9w \in \mathcal{G}, \end{cases}$$

which yields $(0,0,0,0)$, a multiple of v_2.

For λ_3 :

$$A + 8I = \begin{pmatrix} 10 & 10 & 9 & - \\ 9 & 8 & - & - \\ - & - & 8 & 9 \\ 9 & - & - & 8 \end{pmatrix},$$

and the tangible value of the third column of its adjoint is

$$v_3 = (25, 26, 27, 26) = 25\,(0, 1, 2, 1)\,.$$

This can also be obtained when multiplying the eigenmatrix by

$$E_{4^{th}\ row + 1\cdot 3^{rd}\ row} E_{4^{th}\ row + 2\cdot 2^{nd}\ row} E_{2^{nd}\ row + 1^{st}\ row} E_{1,4}$$

on the left:

$$\begin{pmatrix} 9 & - & - & 8 \\ 9^{\nu} & 8 & - & 8 \\ - & - & 8 & 9 \\ 11^{\nu} & 10^{\nu} & 9^{\nu} & 10^{\nu} \end{pmatrix},$$

and solving the tropically linear system

$$\begin{cases} 9x + 8w \in \mathcal{G}, \\ 8y + 8w \in \mathcal{G}, \\ 8z + 9w \in \mathcal{G}, \end{cases}$$

which yields $(7, 8, 9, 8) = 7\,(0, 1, 2, 1)$, a multiple of v_3.

For λ_4

$$A + 1I = \begin{pmatrix} 10 & 10 & 9 & - \\ 9 & 1^{\nu} & - & - \\ - & - & 1 & 9 \\ 9 & - & - & 1 \end{pmatrix},$$

and the tangible value of the second column of its adjoint is

$$v_4 = (12, 27, 28, 20) = 12\,(0, 15, 16, 8)\,.$$

This can also be obtained when multiplying the eigenmatrix by

$$E_{4^{th} row + (-1)\cdot 1^{st}\ row} E_{4^{th} row + \cdot 2^{nd}\ row} E_{2^{nd} + (-1)\cdot 1^{st}\ row}$$

on the left:

$$\begin{pmatrix} 10 & 10 & 9 & - \\ 9^\nu & 9 & 8 & - \\ - & - & 1 & 9 \\ 9^\nu & 9^\nu & 8^\nu & 1^\nu \end{pmatrix},$$

and solving the tropically linear system

$$\begin{cases} 10x + 10y + 9z \in \mathcal{G}, \\ 9y + 8z \in \mathcal{G}, \\ 1z + 9w \in \mathcal{G}, \end{cases}$$

which yields $(x, 8, 9, 1)$, where $x \leq 8$.

From the fourth position of $Av \models_{gs} \lambda v$, we get $9x \models_{gs} 2$ which implies $x = -7$. Thus the eigenvector is $(-7, 8, 9, 1) = -7(0, 15, 16, 8)$, a multiple of v_4.

Next, we examine the dependence of the eigenvectors, using the matrix W having these vectors for its columns:

$$W = \begin{pmatrix} 30 & 28 & 25 & 12 \\ 29 & 28 & 26 & 27 \\ 28 & 28 & 27 & 28 \\ 29 & 28 & 26 & 20 \end{pmatrix}.$$

The determinant of W is 112^ν and is obtained by the permutations $(1)(2)(3\ 4)$ and $(1)(2\ 4)(3)$. One can see that the ghost part of the product is attained in the principal sub-matrix $\{2,3,4\} \times \{2,3,4\}$, where the pathology of the index sets occurs. We rewrite W using the eigenvalues and the entries of $A = (a_{i,j})$, in order to understand this dependence:

$$W = \begin{pmatrix} \lambda_1^3 & a_{1,2}\lambda_2^2 & \lambda_3^2 a_{1,3} & \lambda_4^2 a_{1,2} \\ \lambda_1^2 a_{2,1} & \lambda_1\lambda_2^2 & \lambda_3 a_{2,1}a_{1,3} & a_{1,3}a_{3,4}a_{4,1} \\ \lambda_1 a_{3,4}a_{4,1} & a_{3,4}a_{4,1}a_{1,2} & \lambda_3 a_{1,2}a_{2,1} & a_{3,4}a_{4,1}a_{1,2} \\ \lambda_1^2 a_{4,1} & \lambda_2 a_{4,1}a_{1,2} & \lambda_3 a_{4,1}a_{1,3} & \lambda_4 a_{4,1}a_{1,2} \end{pmatrix}.$$

The determinant is attained by

$$\lambda_1^3(\lambda_1\lambda_2^2)(a_{3,4}a_{4,1}a_{1,2})(\lambda_3 a_{4,1}a_{1,3}) \text{ and } \lambda_1^3(a_{1,3}a_{3,4}a_{4,1})(\lambda_3 a_{1,2}a_{2,1})(\lambda_2 a_{4,1}a_{1,2}).$$

All elements are identical except for $\lambda_1\lambda_2$, which appears in the first term, and $a_{1,2}a_{2,1}$, which appears in the second term, but we know that $\lambda_1\lambda_2 = a = a_{1,2}a_{2,1}$.

That is, the determinant is not attained twice because of arbitrarily equal values (for example $1 \cdot 6 = 3 \cdot 4$), or because of repeating values in the entries of A (such as 9 and 10), but rather is attained twice by precisely the same entries of A after a permution:

$$\lambda_1^3(\lambda_1\lambda_2^2)(a_{4,1}a_{1,2}a_{3,4})(a_{4,1}a_{1,3}\lambda_3) =$$

$$\lambda_1^3[a][\lambda_2 a_{4,1}a_{1,2}][a_{3,4}a_{4,1}a_{1,3}][\lambda_3] =$$

$$\lambda_1^3[\lambda_2 a_{4,1}a_{1,2}][a_{3,4}a_{4,1}a_{1,3}][a_{1,2}a_{2,1}\lambda_3].$$

3.3. **Resolving the pathology.** In this section we offer sufficient conditions for independence, and provide two conjectures on the eigenvectors of the quasi-inverse of a matrix.

3.3.1. *The resolution by means of disjoint index sets.*
We show that the pathology is a Zariski-closed condition.

Theorem 3.4. *Let $A = (a_{i,j})$ be an $n \times n$ nonsingular matrix, with tangible characteristic polynomial (coefficient-wise) and n distinct eigenvalues. If A satisfies the difference criterion, then the eigenvectors of A are tropically independent.*

Proof. Let $f_A(x) = \sum_{i=0}^{n} \alpha_i x^{n-i}$ be the characteristic polynomial of A, which means

$$\alpha_0 = 0,\ \alpha_1 = tr(A) \ \text{ and } \ \alpha_n = \det(A).$$

Without loss of generality, we may assume that $tr(A)$ is $a_{1,1}$, i.e.,

$$a_{1,1} > a_{t,t}\ \forall t \neq 1. \tag{3.5}$$

In order to get n distinct eigenvalues, we must have $f_A(x) = f_A^{es}(x)$, or equivalently:

$$\lambda_1 = tr(A) > \lambda_2 = \frac{\alpha_2}{tr(A)} > \lambda_3 = \frac{\alpha_3}{\alpha_2} > \cdots > \lambda_{n-1} = \frac{\alpha_{n-1}}{\alpha_{n-2}} > \lambda_n = \frac{\det(A)}{\alpha_{n-1}}, \tag{3.6}$$

where $\{\lambda_l\}_{l=1}^{n}$ are the corner-roots of f_A. Otherwise $\exists t, s$ such that $f_A(\lambda_t) \in \mathcal{T}$ or $\lambda_t = \lambda_s$, contrary to hypothesis. Thus

$$\lambda_1 \cdots \lambda_k = \alpha_k, \quad \forall k, \quad \text{and in particular} \ \ \lambda_1 \cdots \lambda_n = \det(A), \tag{3.7}$$

where α_k is attained *solely* by this product since $f_A \in \mathcal{T}[x]$.
Moreover, $\{1\} = \mathrm{Ind}_1 \subseteq \mathrm{Ind}_k$, for every $k \geq 1$. Otherwise, $a_{1,1}$ together with α_{k-1} yields a permutation on k indices, dominated by α_k, and we get

$$\lambda_1 = a_{1,1} \leq \frac{\alpha_k}{\alpha_{k-1} \cdot a_{1,1}} \cdot a_{1,1} = \lambda_k,$$

contradicting (3.6).

The indices $\mathrm{Ind}_k \setminus \{1\} = \{j_2, \dots, j_k\}$ are where the pathology lies. Straightforward, since $\mathrm{Ind}_0 = \emptyset$, we have that $\forall s \in \{2, ..., k\}\ \exists i \leq k :\ j_s \in I_{\lambda_i}$. Assume that

$$\forall l \leq k \ \ \mathrm{Ind}_{l-1} \subseteq \mathrm{Ind}_l \ \text{ and } \ \exists s \in \{2, ..., k\} :\ j_s \notin \mathrm{Ind}_{k+1}.$$

The index j_s must appear again in the index set of some subsequent coefficient (since Ind_n of the determinant includes all of the indices). Denoting the first such coefficient as α_t, that means

$$j_s \in \mathrm{Ind}_t \setminus \mathrm{Ind}_{t-1} = I_{\lambda_t},\ t > k,$$

contradicting the difference criterion.

Thus we require that the $\mathrm{Ind}_k \subseteq \mathrm{Ind}_{k+1}$, $\forall k$, and without loss of generality we may assume that $j_k = k$, $\forall k = 1, \dots, n$. As a result, $I_{\lambda_k} = \{k\}$ and (3.5)-(3.6) become:

$$a_{1,1} = \lambda_1 > a_{t,t}, \quad \forall t \neq 1, \tag{3.8}$$

and $\forall k > 1$:

$$a_{1,1} > \lambda_k = \frac{\alpha_k \cdot a_{t,t}}{\alpha_{k-1} \cdot a_{t,t}} \geq a_{t,t}, \quad \forall t \geq k, \tag{3.9}$$

with equality only when $k = t$, and

$$\beta_{k,t} = \max\{\lambda_k, a_{t,t}\}, \quad \forall t < k. \tag{3.10}$$

Thus, the eigenmatrices are (A written off the diagonal means that the off-diagonal entries are identical to those of A):

$$A+\lambda_1 I=\begin{pmatrix} \lambda_1 & & A \\ & \ddots & \\ A & & \lambda_1 \end{pmatrix}, \; A+\lambda_2 I=\begin{pmatrix} \lambda_1 & & & A \\ & \lambda_2 & & \\ & & \ddots & \\ A & & & \lambda_2 \end{pmatrix}, \; A+\lambda_3 I=\begin{pmatrix} \lambda_1 & & & & A \\ & \beta_{3,2} & & & \\ & & \lambda_3 & & \\ & & & \ddots & \\ A & & & & \lambda_3 \end{pmatrix}, \dots$$

Or in general

$$A + \lambda_k I = (b^{(k)}_{i,j}),$$

where

$$b^{(k)}_{i,i} = \begin{cases} \lambda_1, & i = 1 \\ \beta_{k,i}, & 1 < i < k \\ \lambda_k, & i \geq k \end{cases} \tag{3.11}$$

and $b^{(k)}_{i,j} = a_{i,j}, \quad \forall i \neq j$.

Let $\mathrm{adj}(A) = (a'_{i,j})$ and $W = (w_{i,j})$ be the matrix with the (tangible value of the) eigenvectors for its columns.

On the one hand the (k, k) position of $A + \lambda_k I$ is λ_k, and the (k, k) position of

$$(A + \lambda_k I)(\mathrm{adj}(A + \lambda_k I))$$

is

$$\det(A + \lambda_k I) = f_A(\lambda_k) \in \mathcal{G}_{0_\mathcal{F}},$$

attained by

$$\alpha_k \lambda_k^{n-k} + \alpha_{k-1}\lambda_k^{n-k+1} = (\lambda_1 \cdots \lambda_{k-1}\lambda_k^{n-k+1})^\nu.$$

Therefore, the (k, k) position of $\mathrm{adj}(A + \lambda_k I)$ (and therefore of W) is **at most**

$$\lambda_1 \cdots \lambda_{k-1}\lambda_k^{n-k}.$$

On the other hand, one of the summands in the (k, k) position of $\mathrm{adj}(A + \lambda_k I)$ is

$$\lambda_k^{n-k} \det(M_{k,\dots,n}),$$

where $M_{k,\dots,n}$ is the $(k-1)\times(k-1)$-principal sub-matrix of $A+\lambda_k I$, obtained by rows and columns $1, \dots, k-1$. Since it differs from the corresponding $(k-1)\times(k-1)$-principle sub-matrix of A by having a greater diagonal, we get:

$$\det(M_{k,\dots,n}) \geq_\nu \alpha_{k-1} = \lambda_1 \cdots \lambda_{k-1}.$$

Thus, the (k, k) position in $\mathrm{adj}(A + \lambda_k I)$ is **at least**

$$\lambda_1 \cdots \lambda_{k-1}\lambda_k^{n-k}.$$

As a result, the diagonal positions of W are:

$$w_{k,k} = \lambda_1 \cdots \lambda_{k-1}\lambda_k^{n-k}. \tag{3.12}$$

We recall that when A is nonsingular, $\det(\operatorname{adj}(A)) = \det(A)^{n-1}$ is attained solely by the permutation σ^{-1}, where σ attains $\det(A)$.

Since $\det(A) = \lambda_1 \cdots \lambda_n$, we have

$$\det(\operatorname{adj}(A)) = \lambda_1^{n-1} \cdots \lambda_n^{n-1} < \prod_{i=1}^{n} w_{i,i}.$$

Next, using the relations (3.5)-(3.12), we show that each permutation of W weights at least as much as the corresponding permutation in $\operatorname{adj}(A)$, with the unique maximal weight coming from the Id-permutation.

Examining the entries of W, we notice that the diagonal entries of A are replaced according to (3.11), and that λ_n does not appear. Every non-Id-permutation π of W includes cycles of A, and therefore satisfies at least one of the following properties:

- π is a single k-cycle. Then the permutation π of W is smaller than or equal to $\lambda_k = \lambda_1 \cdots \lambda_k \in \mathcal{T}$ (with $\mathrm{Ind}_k = \{1, \dots, k\}$).
- π contains the product of two different k-cycles. Then the permutation π of W is strictly smaller than λ_k^2, because every coefficient is tangible.
- π is a k-cycle that does not include one of the indices $1, \dots, k$. Then the permutation π of W is strictly smaller than λ_k.
- The permutation π of W includes, other than entries of A, only the eigenvalue λ_1. Then the cycles of π do not include the index 1, and since $\lambda_1 = a_{1,1}$ it will expend a k-cycle into a $k+1$-cycle which is strictly smaller than $(\lambda_1 \lambda_2 \cdots \lambda_{k-1} \lambda_k)\lambda_k$.
- The permutation π of W does not include λ_t for any t. Then π is a product of $n-1$ permutations of A, whose product is smaller than or equal to $\det(\operatorname{adj}(A))$.

As a result, every permutation other than Id is strictly smaller than the Id. Thus, the determinant of W is attained solely by the diagonal and therefore is tangible. $\square$

3.3.2. *The resolution by means of the quasi-inverse.*

We examine the implications of this pathology on $\operatorname{adj}(A)$, which according to Theorem 2.14 and Lemma 2.16 has determinant $\det(A)^{n-1}$ attained solely by the permutation σ^{-1}, where $\det(A)$ is attained solely by σ. This will lead us to Conjecture 3.5 and Conjecture 3.6.

Calculating the adjoint of the matrix A from Example 3.3 we get

$$\operatorname{adj}(A) = \begin{pmatrix} - & - & - & 19 \\ - & 27 & - & 27 \\ 19 & 28 & - & 28 \\ - & - & 19 & - \end{pmatrix},$$

with characteristic polynomial

$$f_{\operatorname{adj}(A)}(x) = x^4 + 27x^3 + 47x^2 + 74^{\nu}x + 84,$$

obtained by the permutations (2), $(3\ 4)$, $(2\ 4\ 3)$ and $(2)(3\ 4)$, $(3\ 1\ 4)(2)$, respectively. (It is easy to see that the determinant is attained by the permutation invert to the one attaining $\det(A)$, and taking on its value three times. $f_{A^{\nabla}}$ is obtained by dividing the k^{th} coefficient by $\det(A)^k$ for every k: $f_{A^{\nabla}}(x) = x^4 + (-1)x^3 + (-9)x^2 + (-10^{\nu})x + (-28)$, which satisfies Theorem 5.1 in [6].)

Therefore,

$$\begin{cases} \{2\} \setminus \emptyset = \{2\}, \\ \{3,4\} \setminus \{2\} = \{3,4\}, \\ \{2,3,4\} \setminus \{3,4\} = \{2\}, \\ \{1,2,3,4\} \setminus \{2,3,4\} = \{1\}, \end{cases}$$

respectively. However, calculating the eigenvalues of $\mathrm{adj}(A)$ we discover that

$$f^{es}_{\mathrm{adj}(A)}(x) = x^4 + 27x^3 + 74^\nu x + 84.$$

That is, the dependence in the principal sub-matrix $\{2,3,4\} \times \{2,3,4\}$ (identical to the location of dependence in W) increased the coefficient of x, which caused the monomial of x^2 to be inessential.

Indeed, the dependence in W caused by

$$\lambda_2(\lambda_1\lambda_2)(a_{4,1}a_{1,2}a_{3,4})(a_{4,1}a_{1,3})\lambda_3 = \lambda_2(a_{4,1}a_{1,2})(a_{3,4}a_{4,1}a_{1,3})(a_{1,2}a_{2,1})\lambda_3,$$

would yield the equality

$$(\lambda_1\lambda_2)(a_{4,1}a_{1,2}a_{3,4})(a_{4,1}a_{1,3})\frac{a_{4,1}a_{1,3}a_{3,4}}{a_{1,1}} = (a_{4,1}a_{1,2})(a_{3,4}a_{4,1}a_{1,3})(a_{1,2}a_{2,1})\frac{a_{4,1}a_{1,3}a_{3,4}}{a_{1,1}},$$

using entries of A to express $\lambda_2\lambda_3$ and later $\lambda_1\lambda_2$.

Replacing $\frac{a_{4,1}a_{1,2}}{a_{1,1}}$ by $a_{2,2}$ on both sides, we obtain the equality

$$(a_{1,2}a_{2,1}a_{3,4})(a_{4,1}a_{1,3}a_{3,4})(a_{4,1}a_{1,3}a_{2,2}) = (a_{3,4}a_{4,1}a_{1,2})(a_{2,1}a_{1,3}a_{3,4})(a_{4,1}a_{1,3})a_{2,2}$$

which causes the coefficient of x to be a ghost.

As a result,

$$\begin{cases} I_{\lambda_1} = \{2\} \setminus \emptyset = \{2\}, \\ I_{\lambda_{2,3}} = \{2,3,4\} \setminus \{2\} = \{3,4\}, \\ I_{\lambda_4} = \{1,2,3,4\} \setminus \{2,3,4\} = \{1\} \end{cases},$$

where $\lambda_1 = 27,\ \lambda_{2,3} = 23.5$ (with multiplicity 2), $\lambda_4 = 10$, and independent eigenvectors:
$v_1 = (66, 81, 82, 74) = 66(0, 15, 16, 8)$,
$v_{2,3} = 65^{-1}\underbrace{(65, 69.5, 74, 69.5)}_{\text{from the third column}} = (0, 4.5, 9, 4.5) = 69.5^{-1}\underbrace{(69.5, 74, 78.5, 74)}_{\text{from the fourth column}}$,
and $v_4 = (74, 65, 55, 65) = 55(19, 10, 0, 10)$.

We recall Theorem 2.22 for the following conjecture.

Conjecture 3.5. *Let A be a non-singular matrix with n distinct eigenvalues. If the eigenvectors of A are dependent, then*

- *$det(A)f_{A^\nabla}(x)$ strictly ghost-surpasses $x^n f_A(x^{-1})$.*
- *$f_{A^\nabla} \neq f^{es}_{A^\nabla}$, which means A^∇ has fewer distinct eigenvalues than A.*

Conjecture 3.6. *Let A be a non-singular matrix. If A^∇ has n distinct eigenvalues then their corresponding eigenvectors are independent.*

3.3.3. *The resolution by means of generalized eigenspaces.*

Eigenspaces are studied in [21] and are defined in [22] to be spanned by supertropical eigenvectors. Let $V = F^n$.

Definition 3.7. A tangible vector $v \in V$ is a **generalized supertropical eigenvector** of A, with **generalized supertropical eigenvalue** $\lambda \in \mathcal{T}$, if $(A + \lambda I)^m v$ is ghost for some $m \in \mathbb{N}$. If $A^m v$ is itself ghost for some m, we call the eigenvector v **degenerate**.

The minimal such m is called the **multiplicity** of the eigenvalue (and also of the eigenvector).

The **generalized supertropical eigenspace** V_λ with **generalized supertropical eigenvalue** $\lambda \in \mathcal{T}$ is the set of generalized supertropical eigenvectors with generalized supertropical eigenvalue λ.

Note that if v is a degenerate eigenvector, then it belongs to V_λ for all sufficiently small λ.

Lemma 3.8. *V_λ is indeed a supertropical subspace of V.*

Proof. Let $v, u \in V_\lambda$. Thus $\exists m, t:\ (A + \lambda I)^m v \models_{gs} 0_{\mathcal{F}}$ and $(A + \lambda I)^t u \models_{gs} 0_{\mathcal{F}}$, and therefore for any $a \in \mathcal{F}$

$$(A + \lambda I)^{(m+t)}(v + au) = (A + \lambda I)^t (A + \lambda I)^m v + a(A + \lambda I)^m (A + \lambda I)^t u \models_{gs} 0_{\mathcal{F}}.$$

$\square$

Remark 3.9. We have the following hierarchy:

$Av \models_{gs} \lambda v$, implies $A^m v \models_{gs} \lambda^m v$, implies $A^m v + \lambda^m v \models_{gs} 0_{\mathcal{F}}$, implies $(A+\lambda I)^m v \models_{gs} 0_{\mathcal{F}}$.

This approach gives some insight into the difference criterion.

Lemma 3.10. *For $m = n!$, the diagonal is a dominant permutation of A^m. The difference criterion is satisfied for A iff the diagonal entries of A^m are distinct.*

Proof. The first assertion is clear.

Suppose that in A^m some index i appears in both I_k and $I_{k'}$ for $k < k'$, where k is taken minimal such. Then all the previous I_j are disjoint, so, rearranging the diagonal entries, we may assume that i appears in the $|I_1| + \cdots + |I_{k-1}| + \alpha$ position in the diagonal for some $1 \le \alpha \le |I_k|$. But i must also appear in the analogous position arising from $I_{k'}$, which is further down, so A^m has a double entry, and f_A has a double root. $\square$

Lemma 3.11. *If A is nonsingular and diagonally dominant, then the diagonal of A is tangible.*

Proof. The determinant is the product of the diagonal entries, so each is tangible. $\square$

In view of Remark 2.20, we can refine V_λ. Write $f_A = \prod_i g_i$ where $g_i = (x + \lambda_i)^{t_i}$, with the λ_i distinct, and let $\tilde{f}_i = \prod_{j \ne i} g_j$. (Thus, $f_A = g_i \tilde{f}_i$.) Suppose $v \in \tilde{f}_i(A) V_\lambda$. Then $g_i(A)v \in f_A(A)V$ is ghost, implying $v \in V_\lambda$. Thus, we can define the subspace

$$V'_{\lambda_i} = \left(\prod_{j \ne i} g_j(A) \right) V.$$

Definition 3.12. A matrix A is **strongly nonsingular** if A^m is nonsingular for all m.

In this case, A has no nonzero degenerate eigenvectors, for if $A^m \in (\mathcal{G} \cup \{0_\mathcal{F}\})^n$ then $A^m v \in (\mathcal{G} \cup \{0_\mathcal{F}\})^n$.

Theorem 3.13. *If A is strongly nonsingular, then the V'_λ are independent.*

Proof. We can replace A by $A^{n!}$ and assume that A is diagonally dominant and that V'_λ are eigenspaces of A. We use the notation following Lemma 3.11. Given a ghost dependence, i.e., $\sum_{i=1}^u \gamma_i \tilde{f}_i(A) v_i$ ghost for tangible γ_i, the fact that A is strongly nonsingular implies that $\lambda_u^{t_u}$ dominates all $\lambda_u^{t_u - j}\beta^j$, for all $\beta < \lambda_u$. Hence, when x is to be specialized to these β, $\lambda_u^{t_u}$ dominates $\sum_j \lambda_u^{t_u - j} x_u^j = g_u$, and thus some component of $\gamma_u \lambda_u^{t_u} \tilde{f}_u(A) v_u$ is dominant in $\gamma_u g_u(A) \tilde{f}_u(A) v_u$, a ghost. Therefore $f_u(A) = \tilde{f}_u(A) g_u(A) v_u$ is ghost, implying some power of A ghost annihilates v_u, in contradiction to A being strongly nonsingular. □

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